in the height of high water; that there were reasons to expect some systematic difference between the two calculations, and that all the considerable errors of time fall on those very small rises of water which are of frequent occurrence at Aden.

Two other comparisons were also made, one with the Indian predictions of 1887, and the other with actuality of 1884. In the latter case, when a few very small tides were omitted, the probable error was 7m. in the time, and 1·4 inches in height. It is concluded from these comparisons that, with good values for the tidal constants, the tables lead to excellent predictions, even better than are required for nautical purposes.

It is probable that this method may be applied to ports of second-rate importance, where there are not sufficient data for very accurate determination of the tidal constants. Suggestions are made for very large abridgment of the tables in such cases, accompanied, of course, by loss of accuracy.

The question of how far to go in each case must depend on a variety of circumstances. The most important consideration is likely to be the amount of money which can be expended on computation and printing; and after this will come the trustworthiness of the tidal constants, and the degree of desirability of an accurate tide-table. The aim of the paper was to give the tables in a simple form, and if, as seems certain, the mathematical capacity of an ordinary ship's captain will suffice for the use of the tables, whether in full or abridged, the principal object in view has been attained.

THEORY OF FUNCTIONS.[1]

I.

THE papers in Herr Schwarz's "Gesammelte Abhandlungen" range over a wide field. But varied as the subject-matter of the papers is, most of them have an internal connection which lies in profound study of the theory of functions.

Herr Schwarz is a pupil of Weierstrass, but he has been greatly influenced by the writings of Riemann. Both these eminent mathematicians have developed a theory of functions. Whilst Weierstrass starts in all his investigations with analytical expressions and operations, Riemann, following methods borrowed from the theory of potential, bases his theory on certain partial differential equations which express fundamental properties of functions, and by developing general properties of all functions tries to replace calculation by reasoning. Both applied their theories to the Abelian functions, and here Riemann's investigations proved to be more general. His speculations are, however, of such generality, that objections were raised by Kronecker and Weierstrass as to the validity of his reasoning, and thus it became doubtful whether his most important theorems were actually proved.

In consequence, various mathematicians have attempted to place Riemann's theorems on a more satisfactory basis, to graft Riemann's far-reaching speculations on the more strongly-rooted methods of Weierstrass. These attempts were made, by Neumann and others, in connection with the theory of potential, whilst other mathematicians used purely analytical investigations. Among the latter, Schwarz stands out most prominently, and those of his papers which relate to this subject seem to me to deserve most attention.

As the subject in its great generality and abstractness has received little attention in England, it seems desirable to give an outline of Riemann's theory in order to be the better able to appreciate Schwarz's papers. The notion of a function, originally equivalent to that of a power, was gradually extended till Bernoulli gave a first definition, which was made more concise by Leibnitz. According to them, y is called a function of x if there exists an equation between these variables which makes it possible to calculate y for any given values of x for all (real) values of x from $-\infty$ to $+\infty$. This last condition excludes series which are convergent only for certain values of x. The functions here included will be essentially those which are given by an equation involving a finite number of algebraical, trigonometrical, and logarithmic or exponential terms. Such a function will, as a rule, be continuous—practically the only exceptions being infinite values of y—and it will have a derived function. In fact, these two properties, continuity and existence of a derived function, were considered as identical. Such a function is geometrically represented by a curve, x and y being taken as Cartesian co-ordinates of a point. But not every curve can be taken as representing a function.

The next step forward is chiefly due to the work of Fourier. Physical problems required the study of functions with new characteristics. The initial temperature of a rod, lying in the axis of x, may be supposed arbitrarily given, not for all values of x, but only for those which lie within the interval covered by the rod, and in the trigonometrical series means were found for expressing this arbitrarily given temperature in terms of x. This led Dirichlet to the new definition: y is called a function of x if the values of y are *arbitrarily given*, say by a curve, for all values of x *within a given interval*. This involves a deviation from Bernoulli's definition in two directions: the function is originally not given by a mathematical expression, and not for all values of x; in fact, it exists only as far as it is given. The *expression*, by Fourier's series, is not wanted for the definition of the function; it is needed in order to make the latter amenable to mathematical treatment.

The study of functions thus defined leads to new ideas of possible discontinuities. The curve representing the function may have sharp bends, may even have sudden breaks, the value of y suddenly changing from one value to another, different by a finite quantity. At such points we cannot any longer speak of a derived function, though the expression by Fourier's series is still existing. Further investigations in this direction have clearly established the fact that the existence of a derived function is not at all a necessary consequence of the continuity of the function itself. In connection with this it may be mentioned that Riemann has given a first example of a function which is defined by a process of calculating y from x, such that for every value of x there exists a value of y which has within a finite interval an infinite number of discontinuities, and which therefore has no derived function, though it has an integral. Schwarz gives, in one of his papers, an example of a function which, though continuous, has no derived function.

A much greater revolution in the ideas about function was produced by the introduction of imaginary values of the variables. Cauchy defines, in conformity with the second of the above definitions, that $w = u + iv$ is a function of $z = x + iy$ if for each set of values xy the corresponding values of u and v are given. Here a geometrical representation of the variables becomes very useful, if not absolutely necessary, viz. the variable z is represented by that point z in a plane which has the rectangular co-ordinates x, y; and w by that point in another plane which has the rectangular co-ordinates u, v. This enables us to make use of geometrical conceptions, and thus to become more concise in our language. We shall speak of the value z and of the point z representing this value as equivalent, the one completely determining the other.

If we start with a given value z_0 of z, and vary the latter, then the point z will move from the position z_0, describing any curve. To this value z_0 will correspond a value w_0 of w. If z is varied, w will also vary. Let us give z a small change dz; then the point z will move to a point z infinitely near to z_0. If now the corresponding change dw of w is infinitely small of the same order as dz, then the function w will be continuous. To fix the change dz we need not only know its magnitude, but also its direction. A change in either will alter dw, and therefore also the ratio dw/dz. The limiting value of this fraction, obtained by making the magnitude of dz disappear, will thus depend on the direction of dz, and not on the value z_0 only. In other words dw/dz will not be a function of z in Cauchy's sense; w has, in general, no derived function.

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[1] "Gesammelte Abhandlungen." Von H. A. Schwarz. Two Volumes. Berlin: Julius Springer, 1890.)

If w shall have a derived function, the definition of a function must be narrowed. Geometrically, it is seen at once what the required condition will be: dw/dz shall have the same value for all infinitely small values of dz; hence the magnitude of dw must be proportional to that of dz and independent of the direction of dz. Hence three positions of z near z_0 and the corresponding positions of w must form two similar triangles. Or: If the point z describes any figure, then the point w will describe a figure which in its smallest parts is similar to it.

If this condition is satisfied, then the function is said to be monogen by Cauchy. Weierstrass calls it an analytical function, whilst Riemann and others confine the word function to this case alone. We shall follow Schwarz in using Weierstrass's expression. We may then say that an analytical function establishes a *conform* (*i.e.* similar in the smallest parts) representation in the w-plane of the points in the z-plane; and conversely, if such conform representation between the points in the two planes is given, then also is w determined as an analytical function of z.

It has to be observed that such representation need not extend over the whole planes of w or z. It is also clear that a function as defined by Bernoulli is analytical. The analytical condition for the existence of a derived function is that $\frac{dw}{dy} = i\frac{dw}{dx}$; or, breaking it up in two real equations, that—

$$(1)\qquad \frac{dv}{dx} = -\frac{du}{dy}, \quad \frac{dv}{dy} = \frac{du}{dx}.$$

From these we obtain, by eliminating v,

$$(2)\qquad \frac{du}{dx^2} + \frac{du}{dy^2} = 0, \text{ or } \Delta u = 0.$$

A function u of x and y can therefore be the real part of an analytical function of z only in case that it is a solution of the partial differential equation $\Delta u = 0$. If u is given satisfying this condition, then $\frac{dv}{dx}$ and $\frac{dv}{dy}$ are given by (1), and therefore v itself is known up to a constant. For instance, $u = e^x \cos y$ satisfies (1), and then (2) gives $v = e \ \sin y + C$, and $w = e^z + iC$.

We thus see that w is determined as an analytical function of z if its real part, u, is given as a solution of a certain partial differential equation, and if besides for one value of z the value of v is given in order to determine the remaining arbitrary constant.

Our problem of determining w is therefore reduced to finding solutions of $\Delta u = 0$. The question is, What are necessary and sufficient conditions to determine any special u?

The equation $\Delta u = 0$ is only a special case of Laplace's equation—

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dx^2} = 0,$$

which, on account of its fundamental importance in the theories of potential and of heat, has received a great deal of attention.

If we suppose that u is independent of z, we obtain our equation $\Delta u = 0$. There is thus a close connection between the theory of analytical functions and that of the potential in a plane. The latter has, on account of its concrete meaning, greatly helped the former. In fact, we have here one of the most interesting examples of the reaction of applied on pure mathematics.

Any results obtained for the potential in space can thus by specializing them so as to relate to a plane be applied to the theory of functions. Of such results there is especially one that has to be mentioned. Green has shown that in space the potential due to attracting masses is uniquely determined for all points within a closed surface which incloses no attracting mass if it is known for all points on the surface. His proof, however, is not a mathematical one, but based on the physical meaning of the potential. An analytical proof of the same theorem in a more general form was given by Sir Wm. Thomson. The same theorem, in more or less different form, is found by other authors, and Dirichlet gave it in his lectures on the theory of potential. Riemann, a pupil of Dirichlet's, therefore calls it "Dirichlet's principle," and by this name it is known in Germany, whilst Maxwell has called it Thomson's theorem.

When specified for a plane, the theorem becomes:—There is always one and only one function of x and y which satisfies the equation $\Delta u = 0$, and which is for all values xy within a given area finite and continuous, and which has for points on the boundary of the area arbitrarily given values.

This, in connection with what has been said about the determination of an analytical function, gives the following theorem:—

An analytical function $w = u + iv$ of $z = x + iy$ is uniquely determined for all points within a closed curve if u is arbitrarily given for all points on the curve, whilst v is given for one point within the curve.

The idea of making the partial differential equation $\Delta u = 0$ the foundation of a theory of analytical functions is due to Riemann (in his Inaugural Dissertation, 1851). In his theory of Abelian functions (1857), he gave a fuller account of some special points, and showed in a brilliant example the usefulness of his conceptions.

In this theory the leading theorem is the one just stated. But I have given it in the simplest form possible, leaving out the conditions about continuity and the conditions which must hold if the function w has more values for one value of z. To make it generally applicable, Riemann had first to invent the surfaces known by his name, consisting of a number of sheets spread over the plane of z, which are connected at single points—the branch points. Besides, he had to discuss the connection of that part of the surface at whose boundary u shall have given values, and this led him to what Maxwell has called cyclosis.

The essential part of Riemann's theory is to find criteria which will determine an analytical function by aid of its discontinuities and boundary conditions, and thus uniquely to define a function independent of a mathematical expression. It enables us, for instance, to decide whether a number of conditions which a function has to satisfy, are independent, sufficient, or inconsistent. If a complete set of conditions is given, the theory makes it possible to decide whether a given expression represents the function, or whether two different expressions are identical. According to the old methods, this identity can only be proved by the actual transformation of one expression into the other, and this requires calculations which are generally long and tedious.

Riemann gives in his dissertation only one application, and that a geometrical one, to which we have also to refer. We have already seen that every analytical function w of z determines a conform representation of the points on the w-plane on the z-plane. If we have two plane surfaces, each bounded by a single curve, then the question arises: Is it possible to find a conform representation of the one on the other? and, if so, what further conditions can be imposed? Riemann gives the theorem that this is always possible and in one way only, in such a manner that a pair of given points of the one figure, one on the boundary and one in the interior, correspond to similarly given points in the other figure. He considers the case when the one figure is a circle, and proves the theorem:—Any simply connected plane figure T can be conformly represented on a circle in such a manner that a given point on the boundary of T corresponds to a given point on the circumference of the circle, and a given interior point of T to the centre of the circle, and this can be done in one way only.

This special case includes the general one, for if two surfaces, T and S, have both been represented on the circle, then also is a representation of one on the other determined.

These and similar important theorems about the general theory of analytical functions were proved by Riemann by aid of "Dirichlet's principle" (Thomson's theorem) already mentioned. This is proved by aid of the calculus of variation. It maintains that a certain definite integral must have necessarily one, and only one, minimum value; its proof, therefore, depends on the calculus of variation.

Whilst the notion and definition of a function were gradually extended, more has also become known about possible discontinuities; and here, again, the theory of potential has greatly helped. In the latter, discontinuities occur which are due to the essentially discontinuous distribution of matter, and which extend over surfaces and lines. Accordingly, we see that an analytical function of z may have discontinuities which extend over lines. Here it must be borne in mind that we know nothing about the possible discontinuities of functions, excepting what we have learned from known functions. Riemann's theory requires that the discontinuities are given, but does not teach us which are possible. Other speculations have shown that the existence of a derived function is not a consequence of continuity; that a function may be integrable without being differentiable, and so on. In fact, of an analytical function in its generality we know almost nothing, and, above all, we do not know how far the methods of the infinitesimal calculus and of the calculus of variations can safely be applied to an unknown analytical function in all its generality. Hence, if, as in the proof of Dirichlet's principle, these methods are used, the functions are endowed with properties which themselves require to be proved. Thus the validity of that principle becomes doubtful, and with it the whole of Riemann's theory. Objections of this kind, first raised by Kronecker and Weierstrass in their lectures, have since been repeated in more specific terms by various mathematicians, and it has long been generally accepted that Riemann's theorems cannot be considered as proved by him.

O. HENRICI.

(*To be continued.*)

HUET'S ANEMOMETER.

THIS instrument is not a new invention. We claim for it rather the honour of being the first of its kind invented. In the article on "Anemometers" in the "Encyclopædia Britannica" mention is made of the efforts of several scientific men in this direction, but neither in this nor in any other such publication can we find any notice of M. Huet's invention. In the said article these instruments are divided into various classes according to the principle upon which they are based, the class to which M. Huet's anemometer would be assigned being described as instruments which "measure the wind force by the difference of level it is capable of producing in an inverted syphon, or U tube, containing water or some other liquid. Lind's anemometer, invented in 1775, is the best known of this type, and is still in common use." Turning to the Philosophical Transactions of the Royal Society for 1775, we find on p. 353 a "'Description and Use of a Portable Wind Gage by Dr. James Lind, Physician, at Edinburgh.' Redde May 11, 1775." Dr. Lind's description of his instrument may be briefly stated as follows:—"This simple instrument consists of two glass tubes . . . connected together like a siphon by a small bent glass tube. . . . The whole instrument is easily turned round upon the spindle by the wind, so as always to present the mouth of the tube towards it. The force or momentum of the wind may be ascertained by the assistance of this instrument by filling the tubes half full of water . . and observe how much the water is depressed by it in the one leg and how much it is raised in the other."

Now we maintain on behalf of M. Huet's anemometer that it exactly answers this description, and, moreover, that his discovery was given to the world half a century earlier than that of Dr. Lind. It is true that Dr. Lind's description and diagram are much more scientific and business-like than those of M. Huet, nevertheless the principle of the instrument is precisely the same.

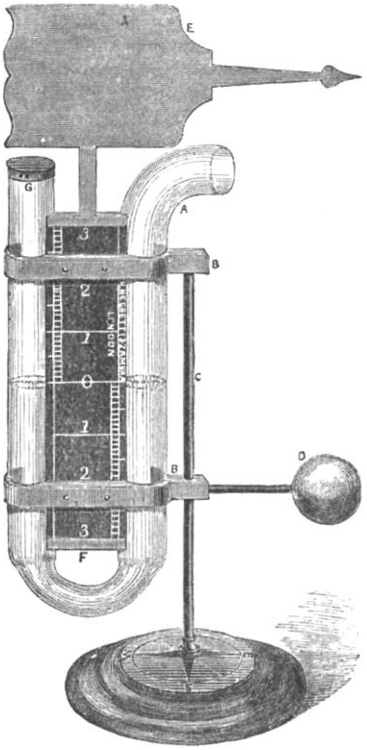

FIG. 1.—Lind's Anemometer.

Translated, M. Huet's description reads as follows:—

"We have worked with success of late to know exactly the quality of the air, its heat, its humidity and its weight by means of the thermometer, the hygrometer, and the barometer, which is a balance of air. But although we have endeavoured to weigh the air, we have not thought of weighing the wind! I made a suggestion about it to Hubin, an excellent English maker of this kind of instrument. He laughed at it, as of a thing easy to think about, but impossible to execute. I gave him the description of an instrument which I had imagined proper to this effect, and he was so satisfied with it that he left me with the design of making it as soon as possible, but his death frustrated his intention. Here it is in few words.